

# Condensate fraction in metallic superconductors and ultracold atomic vapors

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## Abstract

We investigate the condensate density and the condensate fraction of conduction electrons in weak-coupling superconductors by using the BCS theory and the concept of off-diagonal-long-range-order. We discuss the analytical formula of the zero-temperature condensate density of Cooper pairs as a function of Debye frequency and energy gap, and calculate the condensate fraction for some metals. We study the density of Cooper pairs also at finite temperature showing its connection with the gap order parameter and the effects of the electron-phonon coupling. Finally, we analyze similarities and differences between superconductors and ultracold Fermi atoms in the determination of their condensate density by using the BCS theory.

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## 1 Introduction

The condensate fraction of fermionic alkali-metal atoms has been recently investigated [1, 2, 3, 4] by using extended BCS (EBCS) equations [5, 6, 7, 8, 9, 10] from the BCS regime of Cooper-pairs to the BEC regime of molecular dimers [1, 2, 3]. In particular, we have found [1] a remarkable agreement between this simple mean-field theory and the experimental results [11, 12]. These results indicate the presence of a relevant fraction of condensed pairs of

${}^6\text{Li}$  atoms also on the BCS side of the Feshbach resonance. Monte Carlo calculations [13] have shown that the zero-temperature mean-field predictions [1, 2] slightly overestimate the condensed fraction of Fermi pairs. Very recently it has been reported [14] an accurate measurement of the temperature dependence of the condensate fraction for a fermion pair condensate of  ${}^6\text{Li}$  atoms near the unitarity limit of the BCS-BEC crossover. Also these new experimental data [14] are in agreement with mean-field theoretical predictions at finite temperature [3].

In superfluids made of ultracold atoms, the inter-atomic interaction is attractive for all fermions of the system [6]. On the contrary, in metallic superconductors there is an attractive interaction between fermions only near the Fermi surface [15, 16]. As a consequence, the condensate fraction of metallic superconductors has distinctive properties with respect to those of atomic superfluids. Despite the BCS theory is 52 years old [17], the condensate fraction of Cooper pairs in superconductors has been considered only in few papers [18, 20, 19, 21] and in the recent book of Leggett [15]. In fact, in superconductors the condensate fraction has never been measured: only very recently Chakravarty and Kee have proposed to measure it by using magnetic neutron scattering [21].

In this paper we analyze in detail the condensate density of conduction electrons in weak-coupling superconductors at zero and finite temperature by using BCS theory [17] and the concept of off-diagonal-long-range-order [18, 22]. For the first time, we calculate explicitly the density of electronic Cooper pairs and the condensate fraction for various metals and show its dependence on the Debye frequency, the electron-phonon interaction and the energy gap. Another novelty of this paper is the analytical and numerical investigation of the temperature dependence of the condensate fraction, for which we find a power-law behavior. Finally, we compare of the BEC theory of superconductors with the extended BEC theory of ultracold Fermi atoms for obtaining the condensate density and the condensate fraction.

## 2 BCS theory and ODLRO

The BCS Lagrangian density of conduction electrons with spin  $\sigma = \uparrow, \downarrow$  near the Fermi surface is given by

$$\hat{\mathcal{L}} = \sum_{\sigma} \hat{\psi}_{\sigma}^{\dagger} \left( i\hbar \frac{\partial}{\partial t} - \epsilon(\nabla) + \mu \right) \hat{\psi}_{\sigma} + g \hat{\psi}_{\uparrow}^{\dagger} \hat{\psi}_{\downarrow}^{\dagger} \hat{\psi}_{\downarrow} \hat{\psi}_{\uparrow}, \quad (1)$$

where  $\hat{\psi}_\sigma(\mathbf{r}, t)$  is the electronic field operator which satisfies the familiar equal-time anti-commutation rules of fermions. Here  $\epsilon(\nabla)$  is the differential operator such that  $\epsilon(\nabla)e^{i\mathbf{k}\cdot\mathbf{r}} = \epsilon_k e^{i\mathbf{k}\cdot\mathbf{r}}$ , where  $\epsilon_k$  is the energy spectrum of conduction electrons in a specific metal [23]. The attractive interaction between electrons is described by a contact pseudo-potential of strength  $g$  ( $g > 0$ ). For metals this electron-phonon interaction strength is attractive only for conduction electrons near the Fermi surface [15, 16, 17]. The chemical potential  $\mu$  fixes the number  $N$  of conduction electrons.

The Heisenberg equation of motion of the field operator  $\hat{\psi}_\uparrow(\mathbf{r}, t)$  can be immediately derived and reads

$$i\hbar \frac{\partial}{\partial t} \hat{\psi}_\uparrow = [\epsilon(\nabla) - \mu] \hat{\psi}_\uparrow - g \hat{\psi}_\downarrow^\dagger \hat{\psi}_\downarrow \hat{\psi}_\uparrow. \quad (2)$$

In the BCS theory the interaction term of Eq. (2) can be treated within the minimal mean-field approximation  $\hat{\psi}_\downarrow^\dagger \hat{\psi}_\downarrow \hat{\psi}_\uparrow \simeq \hat{\psi}_\downarrow^\dagger \langle \hat{\psi}_\downarrow \hat{\psi}_\uparrow \rangle$ . In this way Eq. (2) becomes

$$i\hbar \frac{\partial}{\partial t} \hat{\psi}_\uparrow = [\epsilon(\nabla) - \mu] \hat{\psi}_\uparrow - \Delta \hat{\psi}_\downarrow^\dagger, \quad (3)$$

where

$$\Delta(\mathbf{r}, t) = g \langle \hat{\psi}_\downarrow(\mathbf{r}, t) \hat{\psi}_\uparrow(\mathbf{r}, t) \rangle \quad (4)$$

is the gap function. The condensate wave function of Cooper pairs [15, 19] is instead given by

$$\Xi(\mathbf{r}, \mathbf{r}', t) = \langle \hat{\psi}_\downarrow(\mathbf{r}, t) \hat{\psi}_\uparrow(\mathbf{r}', t) \rangle. \quad (5)$$

As shown by Yang [18], this two-particle wave function is strictly related to the largest eigenvalue  $N_0$  of two-body density matrix of the system.  $N_0$  gives the number of Fermi pairs in the lowest state, i.e. the condensate number of Fermi pairs [15, 19, 18], and it can be written as

$$N_0 = \int |\Xi(\mathbf{r}, \mathbf{r}', t)|^2 d^3\mathbf{r} d^3\mathbf{r}'. \quad (6)$$

A finite value for the condensate fraction  $f = N_0/(N/2)$  in the thermodynamic limit  $N \rightarrow \infty$  implies off-diagonal-long-range-order [18, 22].

### 3 Gap equation and condensate density

To investigate the properties of the condensate fraction of electronic pairs we adopt the following Bogoliubov representation of the field operator

$$\hat{\psi}_{\uparrow}(\mathbf{r}, t) = \sum_{\mathbf{k}} \left( \frac{u_k}{V^{1/2}} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)} \hat{b}_{\mathbf{k}\uparrow} - \frac{v_k}{V^{1/2}} e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)} \hat{b}_{\mathbf{k}\downarrow}^{\dagger} \right) \quad (7)$$

in terms of the anti-commuting quasi-particle Bogoliubov operators  $\hat{b}_{\mathbf{k}\sigma}$ , with  $V$  the volume of the system and  $E_k = \hbar\omega_k$  the excitation energies of quasi-particles [15, 16].

The thermal averages of quasi-particle Bogoliubov operators are given by

$$\langle \hat{b}_{\mathbf{k}\sigma}^{\dagger} \hat{b}_{\mathbf{k}'\sigma'} \rangle = \frac{1}{e^{\beta E_k} + 1} \delta_{\mathbf{k}\mathbf{k}'} \delta_{\sigma\sigma'} = \bar{n}_k \delta_{\mathbf{k}\mathbf{k}'} \delta_{\sigma\sigma'} , \quad (8)$$

where  $\beta = 1/(k_B T)$  with  $k_B$  the Boltzmann constant,  $T$  the absolute temperature, and  $\bar{n}_k$  is the thermal Fermi distribution.

By using these results, the gap function, Eq. (4), becomes

$$\Delta = \frac{g}{V} \sum'_{\mathbf{k}} (1 - 2\bar{n}_k) u_k v_k , \quad (9)$$

while the condensate number of conduction electrons, Eq. (6), satisfies this expression [1, 19]

$$N_0 = \sum'_{\mathbf{k}} (1 - 2\bar{n}_k)^2 u_k^2 v_k^2 . \quad (10)$$

The 'prime' restricts the summation to states within a shell of width  $\hbar\omega_D$  about the Fermi surface.

To determine the amplitudes  $u_k$  and  $v_k$  of quasi-particles, one inserts Eq. (7) into Eq. (3) and obtains the familiar Bogoliubov-de Gennes equations, which give

$$u_k^2 = \frac{1}{2} \left( 1 + \frac{\xi_k}{E_k} \right) , \quad v_k^2 = \frac{1}{2} \left( 1 - \frac{\xi_k}{E_k} \right) , \quad (11)$$

where

$$\xi_k = \epsilon_k - \mu , \quad E_k = \sqrt{\xi_k^2 + \Delta^2} . \quad (12)$$

Eqs. (9) and (10) can then be written as

$$\Delta = \frac{g}{V} \sum'_{\mathbf{k}} \frac{\Delta}{2E_k} \tanh\left(\frac{\beta E_k}{2}\right) \quad (13)$$

$$N_0 = \sum'_{\mathbf{k}} \frac{\Delta^2}{4E_k^2} \tanh^2\left(\frac{\beta E_k}{2}\right) \quad (14)$$

where  $\tanh(\beta E_k/2) = 1 - 2\bar{n}_k$ .

In the thermodynamic limit, where the volume  $V$  goes to infinity,  $\sum_{\mathbf{k}}$  can be replaced by  $V \int d^3\mathbf{k}/(2\pi)^3 = V \int N(\xi) d\xi$  with  $N(\xi) = \int d^3\mathbf{k}/(2\pi)^3 \delta(\xi - \xi_k)$ . In metals the condition  $\hbar\omega_D \ll \mu$  is always satisfied [16], consequently we can use the approximation  $\int N(\xi) d\xi \simeq N(0) \int d\xi$ , where

$$N(0) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \delta(\mu - \epsilon_{\mathbf{k}}) \quad (15)$$

is the density of states at the Fermi surface. In this way the previous equations (13) and (14) become

$$\frac{1}{gN(0)} = \int_0^{\hbar\omega_D} \frac{\tanh(\frac{\beta}{2}\sqrt{\xi^2 + \Delta^2})}{\sqrt{\xi^2 + \Delta^2}} d\xi \quad (16)$$

$$n_0 = \frac{1}{2} N(0) \Delta^2 \int_0^{\hbar\omega_D} \frac{\tanh^2(\frac{\beta}{2}\sqrt{\xi^2 + \Delta^2})}{\xi^2 + \Delta^2} d\xi \quad (17)$$

where  $n_0 = N_0/V$  is the density of electrons in the condensate.

### 3.1 Zero-temperature condensate

Let us consider first the zero-temperature case ( $T = 0$ ). From Eqs. (16) and (17) we get the zero temperature energy gap  $\Delta(0)$ :

$$\frac{1}{gN(0)} = \ln \left( \frac{\hbar\omega_D}{\Delta(0)} + \sqrt{1 + \frac{\hbar^2\omega_D^2}{\Delta(0)^2}} \right), \quad (18)$$

and the zero-temperature condensate density:

$$n_0(0) = \frac{1}{2} N(0) \Delta(0) \arctan\left(\frac{\hbar\omega_D}{\Delta(0)}\right). \quad (19)$$

This expression shows that the condensate density  $n(0)$  can be expressed in terms of density of states  $N(0)$ , energy gap  $\Delta(0)$  and Debye energy  $\hbar\omega_D$ . Finally, the zero-temperature condensate fraction  $f(0) = n_0(0)/(n/2)$  is given by

$$f(0) = \frac{1}{2} \frac{N(0)}{n} \Delta(0) \arctan\left(\frac{\hbar\omega_D}{\Delta(0)}\right), \quad (20)$$

where  $n$  is the density of conduction electrons.

Under the condition  $\Delta(0) \ll \hbar\omega_D$ , from Eqs. (18) we find the familiar weak-coupling BCS result

$$\Delta(0) = 2\hbar\omega_D \exp\left(-\frac{1}{gN(0)}\right) \quad (21)$$

for the energy-gap order parameter, while the condensate density (19) can be written as [21, 15]:

$$n_0(0) = \frac{\pi}{4}N(0)\Delta(0) \quad (22)$$

We stress that in many real superconductors the simple BCS theory reported above is not accurate, and one has to take into account the retarded electron-electron interaction via phonons [26] and also the Coulomb repulsion [27]. The results obtained above by using the mean-field BCS theory are reliable only in the weak-coupling regime, i.e. for  $\Delta(0) \ll \hbar\omega_D$ , where  $gN(0) \leq 0.3$ . Therefore we will continue our analysis of the condensate fraction only for a class of superconductors which satisfy this condition.

Within the free-electrons Sommerfeld approximation, where the energy spectrum  $\epsilon_k$  of conduction electrons has the simple quadratic behavior  $\epsilon_k = \hbar^2 k^2 / (2m^*)$ , the free particle density of states  $N_{free}(0)$  is related to the total density of conduction electrons by  $n = 4N_{free}(0)\mu/3$  and the zero-temperature condensate fraction reads  $f(0) = 3\pi\Delta(0)/(8\mu)$ . To get a better estimate, we correct the free electron value of  $N(0)$  by an effective mass as obtained from specific heat measurements, i.e. we use the expression  $N(0) = (m^*/m)N_{free}(0)$ .

In the first two columns of Tab. 1 we show the zero-temperature condensate density  $n_0$  and condensate fraction  $f(0)$  of simple metals obtained from Eqs. (19) and (20) by using the experimental data of  $\Delta(0)$  and  $\omega_D$  obtained from Ref. 24 (when the comparison is possible, they agree within a few percent with those reported in Ref. 25). In the third column we report the electron-phonon strength  $gN(0)$  calculated with Eq. (18) knowing  $f(0)$ . The table shows that indeed these simple metals are all in the weak-coupling regime. For completeness, in the forth column we insert the theoretical determination (see Eq. (25)) of the critical temperature  $T_c$ , which is very reliable for these simple metals, when compared with the experimental data.

	$n_0(0)$ [ $10^{-33}$ m $^{-3}$ ]	$f(0)$ [ $10^{-5}$ ]	$gN(0)$	$T_c$ [K]
Cd	4.18	0.9	0.179	0.51
Zn	7.72	1.2	0.172	0.79
Al	22.4	2.5	0.168	1.15
Tl	31.0	5.9	0.263	2.43
In	62.1	10.8	0.267	3.46
Sn	62.5	8.4	0.254	3.68

Table 1: BCS predictions for weak-coupling superconducting metals:  $n_0$  is the zero-temperature condensate density, obtained with Eq. (22);  $f(0) = n_0(0)/(n/2)$  is the zero-temperature condensate fraction;  $gN(0)$  is the electron-phonon strength, calculated with Eq. (18).  $T_c$  is the critical temperature from Eq. (25).

### 3.2 Finite-temperature condensate

Let us now investigate the behavior of the condensate density  $n_0$  at finite temperature  $T$ . Under the condition  $\hbar\omega_D \gg k_B T_c$ , which is always satisfied, near  $T_c$  the energy gap goes to zero according to the power law [16, 15]

$$\Delta(T) = 3.06 k_B T_c \left(1 - \frac{T}{T_c}\right)^{1/2}. \quad (23)$$

Instead for the condensate density  $n_0(T)$ , from Eq. (17) and the previous expression, we find near  $T_c$

$$n_0(T) = 0.43 N(0) \frac{\Delta(T)^2}{k_B T_c} = 4.03 N(0) k_B T_c \left(1 - \frac{T}{T_c}\right). \quad (24)$$

For a generic temperature  $T$  we solve numerically Eqs. (16) and (17). The theoretical critical temperature  $T_c$ , obtained from Eq. (16) setting  $\Delta(T_c) = 0$ , is given by the well-known result [16]

$$k_B T_c = 1.13 \hbar\omega_D \exp\left(-\frac{1}{gN(0)}\right). \quad (25)$$

For simple metals the theoretical critical temperature  $T_c$ , reported in the last column of Table 1, is in good agreement with the experimental one  $T_c^{exp}$ : the

relative difference  $(T_c^{exp} - T_c)/T_c^{exp}$  is not large (i.e. within 10%), and for some metals (Tl, In, Sn) it is quite small (i.e. within 2%).

Taking into account Eq. (20), the BCS theory predicts that the zero-temperature condensate density in superconductors in the weak-coupling regime, can be written as

$$n_0(0) = 1.39 N(0) k_B T_c . \quad (26)$$

This equation resembles the familiar BCS result  $\Delta(0) = 1.764 k_B T_c$  for the zero-temperature energy gap.

We stress that the predictions of the BCS theory can be surely improved by using the Eliashberg theory [25, 26]. This more sophisticated approach will not change the order of magnitude of the numbers in the first two columns of Tab. 1 but it could change the last significant figure.

Coming back to the study of finite-temperature effects, in the upper panel of Fig. 1 we plot the condensate density  $n_0(T)$  vs electron-phonon strength  $gN(0)$  for different values of the temperature  $T$ . As expected, by increasing the temperature  $T$  it is necessary to increase the strength  $gN(0)$  to get the same condensate density.

As it happens for the energy gap  $\Delta(T)$ , one may show that Eqs. (17) and (17) together also imply that the condensate density may be written as its value at  $T = 0$  times a universal function of  $T/T_c$ . In the lower panel of Fig. 1 we plot the condensate density  $n_0(T)/n_0(0)$  as a function of the temperature  $T/T_c$ : in the full range of temperatures the numerical results (solid line) are reasonably well approximated by (dashed line)

$$n_0(T) = n_0(0) \left( 1 - \left( \frac{T}{T_c} \right)^\alpha \right) , \quad (27)$$

with  $\alpha = 3.16$  (best fit).

## 4 Superconductors vs ultracold atoms

In metallic superconductors there is an attractive interaction between fermions only near the Fermi surface [15, 16]. On the contrary, as remarked in the introduction, in superfluid ultracold two-component Fermi atoms, the effective inter-atomic interaction can be made attractive for all atoms of the system by using the technique of Fano-Feshbach resonances [12, 15, 28]. This implies that in the BCS equations for ultracold atoms there is not a natural



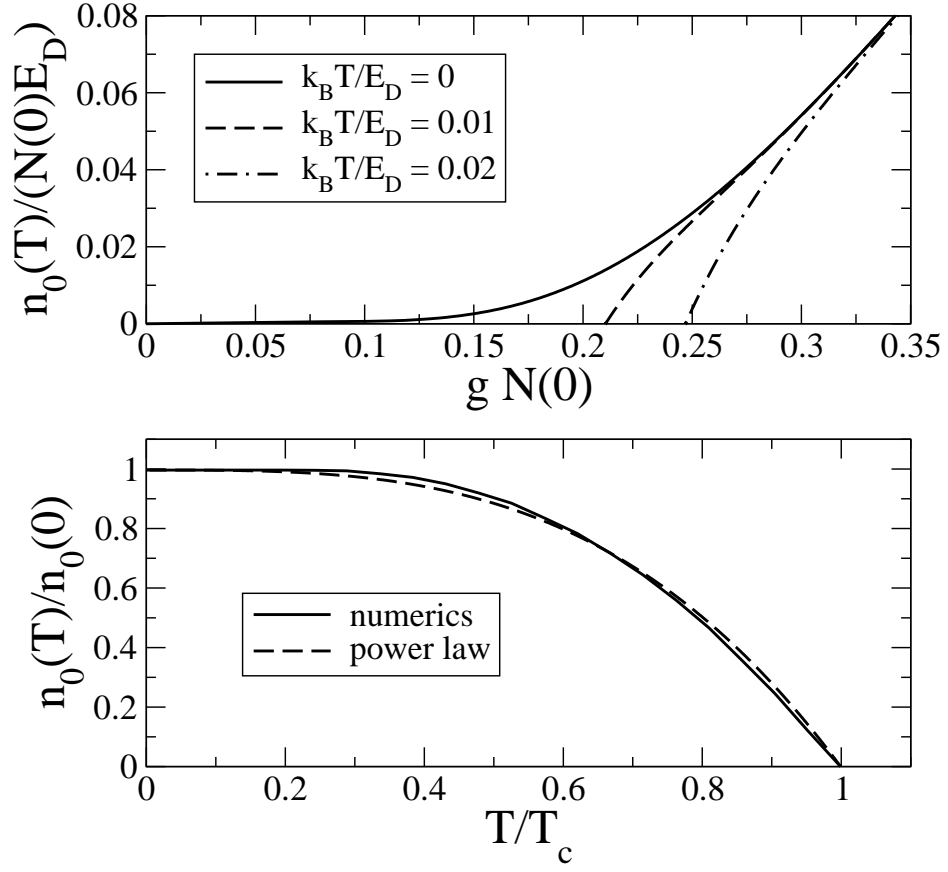


Figure 1: Upper panel: condensate density  $n_0(T)$  vs electron-phonon strength  $gN(0)$  in a superconductor for different values of the temperature  $T$ , where  $N(0)$  is the density of states and  $E_D = \hbar\omega_D$  is the Debye energy. Lower panel: condensate density  $n_0(T)$  as a function of the temperature  $T$  in a superconductor with  $gN(0) = 0.2$ . Solid line: numerical solution of Eqs. (16) and (17); dashed line: analytical approximation, Eq. (27).

ultraviolet cutoff. For attractive ultracold atoms the mean-field BCS theory is given by the gap equation (9) and the number equation

$$N = \sum_{\mathbf{k}} (v_k^2 + 2(u_k^2 - v_k^2)\bar{n}_k) , \quad (28)$$

while the condensate fraction is given by Eq. (10). But, for ultracold atoms, in these equations the sum over momenta is no more restricted within a thin shell around the Fermi surface. As well known, due to the choice of a contact potential, the gap equation (9) diverges in the ultraviolet. This divergence is logarithmic in two dimensions (2D) and linear in three dimensions (3D).

In 3D, a suitable regularization is obtained by introducing the inter-atomic scattering length  $a_F$  via the equation

$$\frac{m}{4\pi\hbar^2 a_F} = -\frac{1}{g} + \frac{1}{V} \sum_{\mathbf{k}} \frac{1}{2\epsilon_k} , \quad (29)$$

where  $\epsilon_k = \hbar^2 k^2 / (2m)$  with  $m$  the atomic mass, and then subtracting this equation from the gap equation [5, 6, 7]. In this way one obtains the 3D regularized gap equation

$$-\frac{m}{4\pi\hbar^2 a_F} = \frac{1}{V} \sum_{\mathbf{k}} \left( \frac{\tanh(\beta E_k/2)}{2E_k} - \frac{1}{2\epsilon_k} \right) . \quad (30)$$

In 2D, quite generally the bound-state energy  $\epsilon_B$  exists for any value of the interaction strength  $g$  between two atoms [10, 29]. For the contact potential the bound-state equation is

$$\frac{1}{g} = \frac{1}{V} \sum_{\mathbf{k}} \frac{1}{\frac{\hbar^2 k^2}{2m} + \epsilon_B} , \quad (31)$$

and subtracting this equation from the gap equation [10, 29] one obtains a 2D regularized gap equation

$$\sum_{\mathbf{k}} \left( \frac{1}{\frac{\hbar^2 k^2}{2m} + \epsilon_B} - \frac{\tanh(\beta E_k/2)}{2E_k} \right) = 0 . \quad (32)$$

The number equation (28) and the renormalized gap equation (30) (or Eq. (32) in 2D) are the so-called generalized BCS equations, from which one determines, for a fixed value of the temperature  $T$  and the average number

of atoms  $N$ , the chemical potential  $\mu(T)$  and the gap energy  $\Delta(T)$  as a function of the scattering length  $a_F$  (or of the bound-state energy  $\epsilon_B$  in 2D). The extended BCS equations can be applied in the full crossover from weak coupling to strong-coupling [15, 28]. In 3D, the crossover is from the BCS state of weakly-interacting Cooper pairs (with  $1/a_F \ll -1$ ) to the Bose-Einstein Condensate (BEC) of molecular dimers (with  $1/a_F \gg 1$ ) across the unitarity limit ( $1/a_F = 0$ ) [6]. In 2D, there is a similar BCS-BEC crossover by increasing the value  $\epsilon_B$  of the bound-state energy [10, 4].

At zero-temperature, by using the continuum limit  $\sum_{\mathbf{k}} \rightarrow V/(2\pi)^3 \int d^3\mathbf{k} \rightarrow V/(2\pi^2) \int k^2 dk$ , the 3D condensate density (10) has a simple analytical expression [1]. The 3D density of states is  $N(\xi) = (2m/\hbar^2)^{3/2} \sqrt{\xi + \mu}/(4\pi^2)$  and the 3D condensate density is given by

$$n_0(0) = \frac{m^{3/2}}{8\pi\hbar^3} \Delta(0)^{3/2} \sqrt{\frac{\mu(0)}{\Delta(0)} + \sqrt{1 + \frac{\mu(0)^2}{\Delta(0)^2}}}. \quad (33)$$

In the 3D BCS regime ( $1/a_F \ll -1$ ), where  $\mu(0)/\Delta(0) \gg 1$  and the size of weaklybound Cooper pairs exceeds the typical interparticle spacing  $k_F^{-1}$ ,  $\mu(0)$  approaches the non-interacting Fermi energy  $\epsilon_F = \hbar^2 k_F^2/(2m)$  with  $k_F = (3\pi^2 n)^{1/3}$  and there is an exponentially small energy gap  $\Delta(0) = 8e^{-2}\epsilon_F \exp(\pi/(2k_F a_F))$ . In this weak-coupling regime the 3D condensate density becomes [1]

$$n_0(0) = \frac{1}{\pi} N(0) \Delta(0) = \frac{3\pi}{2e^2} n \exp\left(\frac{\pi}{2k_F a_F}\right). \quad (34)$$

Notice that this formula is similar to Eq. (22) of weak-coupling superconductors (here  $a_F < 0$ ), but the behavior of  $\Delta(0)$  is quite different.

In 2D, the density of states is constant and reads  $N(\xi) = N(0) = (2m/\hbar^2)/(4\pi)$ . The zero-temperature 2D condensate density is easily obtained[4] as

$$n_0(0) = \frac{1}{4} N(0) \Delta(0) \left( \frac{\pi}{2} + \arctan\left(\frac{\mu(0)}{\Delta(0)}\right) \right), \quad (35)$$

while the zero-temperature 2D energy gap is given by the implicit formula

$$\Delta(0) = 2\epsilon_F \left( \sqrt{1 + \frac{\mu(0)}{\Delta(0)}} - \frac{\mu(0)}{\Delta(0)} \right). \quad (36)$$

From these equations, in the 2D BCS regime ( $0 \leq \epsilon_B \ll \epsilon_F$ ) where  $\mu(0)/\Delta(0) \gg 1$  one finds exactly Eq. (22), but here the energy gap  $\Delta(0)$  depends on the Fermi energy  $\epsilon_F$  and the bound-state energy  $\epsilon_B$  according to the formula [29]

$$\Delta(0) = \sqrt{2\epsilon_F\epsilon_B} , \quad (37)$$

while the chemical potential is  $\mu(0) = \epsilon_F - \epsilon_B/2$ . It is not surprising that in the BCS regime the condensate density of 2D superfluid atoms is formally equivalent to the Eq. (22) we have found for weak-coupling superconductors. In fact, to obtain Eq. (22) we have used the approximation  $\int N(\xi)d\xi \simeq N(0) \int d\xi$  that is exact in the strictly 2D case, and the condition  $\Delta(0) \ll \hbar\omega_D$  which implies that the upper limit of integration is practically  $+\infty$ .

In the previous section we have shown that the BCS equations can be used to determine the (quite small) condensate fraction of superconductors only in the weak-coupling regime. Instead, the extended BCS equations have been used in recent papers [1, 2, 3, 4] to get the condensate fraction of ultracold atoms in the full BCS-BEC crossover. The theory predicts that in the crossover the zero-temperature condensate fraction grows from zero to one. Two experiments [11, 14] have confirmed these predictions for the 3D superfluid two-component Fermi gas.

## 5 Conclusions

In this paper we have studied, within the mean-field BCS theory of superconductors, the condensate of electronic Cooper pairs at zero and finite temperature showing the crucial role played by the Debye frequency and by the electron-phonon interaction. We have found that the zero-temperature condensate fraction  $f(0)$  of weak-coupling metals is quite small ( $\simeq 10^{-5}$ ) and the condensate density increases in metals with higher critical temperature  $T_c$ , according to the law  $f(0) = 1.39 N(0)k_B T_c$ , where  $N(0)$  is the density of states at the Fermi energy. As discussed by Chakravarty and Kee [21], the spin-spin correlation function depends significantly on the condensate density and magnetic neutron scattering can provide a direct measurement of the condensate fraction of a superconductor. In the next future our BCS predictions, which are meaningful for weak-coupling superconductors, could be experimentally tested. In the last part of the paper we have shown similarities and differences between metallic superconductors and atomic Fermi

vapors in the determination of the condensate fraction by using the mean-field BCS theory and its extension in the BCS-BEC crossover.

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